Spectrum Shaping Analysis Using Tunable Parameter of Fractional Based Dirichlet Window

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Abstract— **With the increasing need of spectrum, various computational methods and algorithms have been proposed in the literature. Keeping these views and facts of spectrum shaping capability by FRFT based windows we have proposed a closed form solution for Dirichlet window in fractional domain. This may be useful for analysis of different upcoming generations of mobile communication in better ways which are based on OFDM technique. Moreover, it is useful for real-time processing of non-stationary signals.**

Keywords—**FRFT; Dirichlet window; MSLL; HMLW; SLFOR**

I. INTRODUCTION

When we expand the frequency response of any digital filter by means of Fourier series, we get impulse response of the digital filter in the form of coefficients of the Fourier series. But the resultant filter is unrealizable and also its impulse response is infinite in duration. If we directly truncate this series to a finite number of points we have to face with well known Gibbs phenomenon, so we modify the Fourier coefficients by multiplying it with some finite weighing sequence called window functions, w(n). One desirable characteristics of the Fourier transform of most of the window functions comprises of a central or main lobe of small width containing most of its energy. Also its side lobes should decay rapidly as the frequency tends to π. The properties of various window functions and their parameters are discussed in [1]. After applying the window functions the discontinuities in the frequency response of the digital filter become transition bands [2] between values on either side of the discontinuity whose width depends upon the width of the main lobe of the Fourier transform (FT) of the window function. Also the ripples in the side lobes of FT of the window function produces ripples in the resulting frequency response (which is obtained after circular convolving the desired frequency response and that of the FT of the window function), for all the frequencies. Each window function changes the spectrum in a slight different way and must be chosen according to the specific application required. Applications of window functions are found in harmonic analysis for reducing the effect of spectral leakage [1] and hence in digital filter design and in the field of communication.

In 1980, V. Namias [3] gave the concept of fractional Fourier transform (FRFT) which is generalization of the classical Fourier transform (FT). Later [4] interpreted FRFT as a rotation in the time frequency plane. Nowadays FRFT is used almost in every field where FT is applied. Its various properties and applications can be found in [5]-[8]. Use of FRFT in window functions can be found in [9] where Stankovic et al. have analyzed non-stationary signals with the help of window functions. In [10] fractional Kaiser and PC6 window function is used to show that main lobe width of the window function is dependent on a tunable parameter by which it can be controlled. In [11], Dirichlet and generalized Hamming window functions are solved in the fractional domains. Taking the work of [11] one step forward we have re-solved Dirichlet window function in fractional Fourier domain and have presented it after applying a correction factor to it. We have also shown that the main lobe width of the central lobe is dependent on a tunable parameter α, which is also called the angle of the FRFT and is related to the order of the FRFT i.e. by $\alpha = a\frac{\pi}{2}$. Various parameters of the window function are calculated such as half main-lobe width (HMLW), maximum side-lobe level (MSLL), side-lobe fall-off rate(SLFOR), -3dB bandwidth and -6dB bandwidth [1]. MSLL is defined as the peak value of the side-lobes and HMLW is the frequency at which the central lobe drops to MSLL. SLFOR is the measure of the asymptotic decay of the side lobes and is calculated from the log plot.

The rest of the paper is organized as follows. Section II gives us the definition of FRFT followed by section III in which the analysis of Dirichlet window function is presented. In section IV simulation results are plotted and finally in section V conclusive remarks and references given end our paper.

II. FRACTIONAL FOURIER TRANSFORM (FRFT)

FRFT is obtained by rotating the time frequency axes (t,w) by an angle α to get a new set of axes (u,v) and α is related with the order of the transform by α the relation $\alpha = a\frac{\pi}{2}$. FRFT reduces to FT when rotated by an angle $\frac{\pi}{2}$ and also when it is rotated by 2π, it transforms to an identity transform. FRFT is defined in [4] as follows:

$$X_\alpha(u)=\sqrt{\frac{1-j\cot(\alpha)}{2\pi}}.\exp\left(j\frac{u^2}{2}\cot(\alpha)\right)\times \quad \text{if } \alpha \text{ is not a multiple of } \pi \quad (1)$$

$$\int_{-\infty}^{\infty} x(t).\exp\left(j\frac{t^2}{2}\cot(\alpha)\right)\exp(jut\csc(\alpha))dt;$$

$= x(t);$ if α is a multiple of 2π

$= x(-t);$ if $\alpha+\pi$ is a multiple of 2π

III. DIRICHLET WINDOW FUNCTION

Dirichlet window function as given in [12] is expressed in time domain as:

$$w(t)=1;\ |t|\le\frac{1}{2}$$

$$=0\ ;\ \text{else} \qquad (2)$$

Therefore FRFT of the window function is given as

$$W_\alpha(u)=\sqrt{\frac{1-j\cot(\alpha)}{2\pi}}\exp\left(j\frac{u^2}{2}\cot(\alpha)\right)\times$$

$$\int_{-\infty}^{\infty} w(t)\exp\left[j\frac{t^2}{2}\cot(\alpha)-jut\csc(\alpha)\right]dt \qquad (3)$$

Now (3) can be written as

$$W_\alpha(u)=\sqrt{\frac{1-j\cot(\alpha)}{2\pi}}\exp\left(j\frac{u^2}{2}\cot(\alpha)\right)\times$$

$$\exp\left[\left(\frac{j}{2}\cot(\alpha)\right)\left(-u^2\sec^2(\alpha)\right)\right]\int_{-1/2}^{1/2}\exp\left[\frac{j}{2}\cot(\alpha)(t-u\sec(\alpha))^2\right]dt \qquad (4)$$

By substituting *(t-u sec (α))* and changing the limits of the integration in (4),

$$W_\alpha(u)=\sqrt{\frac{1-j\cot(\alpha)}{2\pi}}\exp\left(-\frac{j}{2}u^2\tan(\alpha)\right)\times$$

$$\int_{-(1/2)-u\sec(\alpha)}^{(1/2)-u\sec(\alpha)}\exp\left[\left(\frac{j}{2}\cot(\alpha)\right)p^2\right]dp \qquad (5)$$

Solving (5) one gets

$$W_\alpha(u)=\left[\sqrt{\frac{1-j\cot(\alpha)}{2\pi}}\exp\left(-\frac{j}{2}u^2\tan(\alpha)\right)\times\frac{1}{2}\sqrt{\frac{\pi}{\frac{j\cot(\alpha)}{2}}}\right]$$

$$\times\left[\begin{array}{l} erfi\left[\sqrt{\frac{j}{2}\cot(\alpha)}\left(\frac{1}{2}-u\sec(\alpha)\right)\right]- \\ erfi\left[\sqrt{\frac{j}{2}\cot(\alpha)}\left(-\frac{1}{2}-u\sec(\alpha)\right)\right]\end{array}\right] \qquad (6)$$

where erfi(z) is an entire analytical function of z which is defined in the whole complex z-plane.

Rearranging (6) one gets

$$W_\alpha(u)=\left[\frac{1}{2}\sqrt{\frac{1}{j\cot(\alpha)}-1}\exp\left(-\frac{j}{2}u^2\tan(\alpha)\right)\right]$$

$$\times\left[\begin{array}{l} erfi\left[\sqrt{\frac{j}{2}\cot(\alpha)}\left(\frac{1}{2}-u\sec(\alpha)\right)\right]- \\ erfi\left[\sqrt{\frac{j}{2}\cot(\alpha)}\left(-\frac{1}{2}-u\sec(\alpha)\right)\right]\end{array}\right] \quad (7)$$

Thus, from (7), it can be seen that the FRFT of Dirichlet window function is directly dependent on the FRFT angle α When we compare this solution to that from [11] we see that the term $\sqrt{\frac{j}{2}\cot(\alpha)}^{-1}$ is missing in [11] which is to be multiplied with it. Thus Dirichlet window function is provided with the correct closed form solution in then FRFT domain.

IV. SIMULATION RESULTS

Dirichlet window in time domain is given in Fig. 1 followed by MSLL plots and SLFOR plots in Fig. 2 and Fig. 3 respectively for a particular value of the tunable parameter a. In Fig. 4 and Fig. 5, three dimensional plots of the Dirichlet window for different values of the tunable parameter vs. amplitude and normalized frequency are drawn.

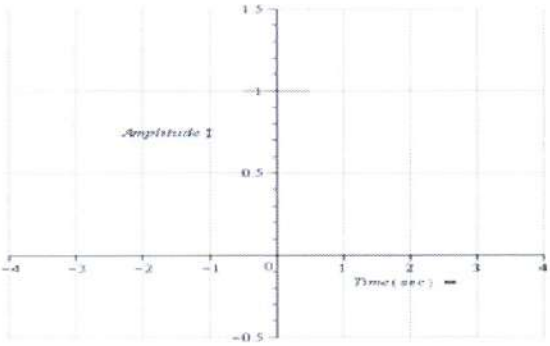

Figure 1. (Time domain Dirichlet window)

Table I comprises of all the tabulated values of MSLL, HMLW, SLFOR, width of the main lobe at -3dB and -6dB down from the peak of the main lobe for different values of a The width of the main lobe limits the frequency resolution of the resultant windowed signal. As the width of the main lobe becomes narrow we are able to distinguish clearly between two adjacent placed frequency components. But on the other side narrowing of the main lobe results in spectral leakage i.e. energy of the window spreads into side lobes so a trade-off is there and the window which is best suited for a particular application is chosen. We see as the value of a decreases from 0.99 to 0.1 the value of MSLL varies from -13.132 dB to -11.898 dB and -3dB bandwidth and -6dB bandwidth decreases from 0.13825 to 0.0206 and 0.18825 and 0.0294 respectively. Side-lobe fall-off rate shows an ir-regular behavior varying from -12.97dB/octave to -19.97dB/octave as a is decreased.

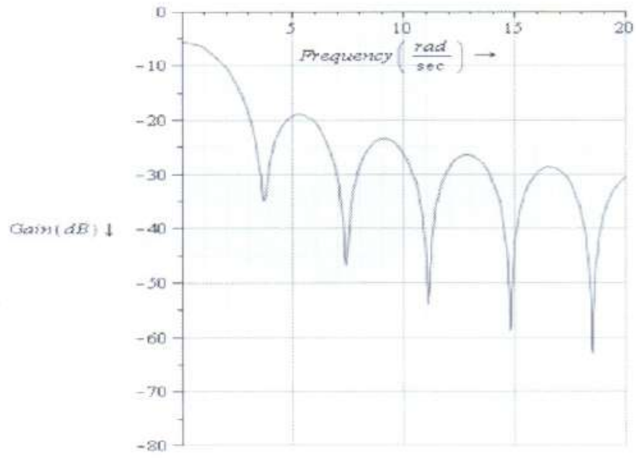

Figure 2. (MSLL Plot for a =0.7)

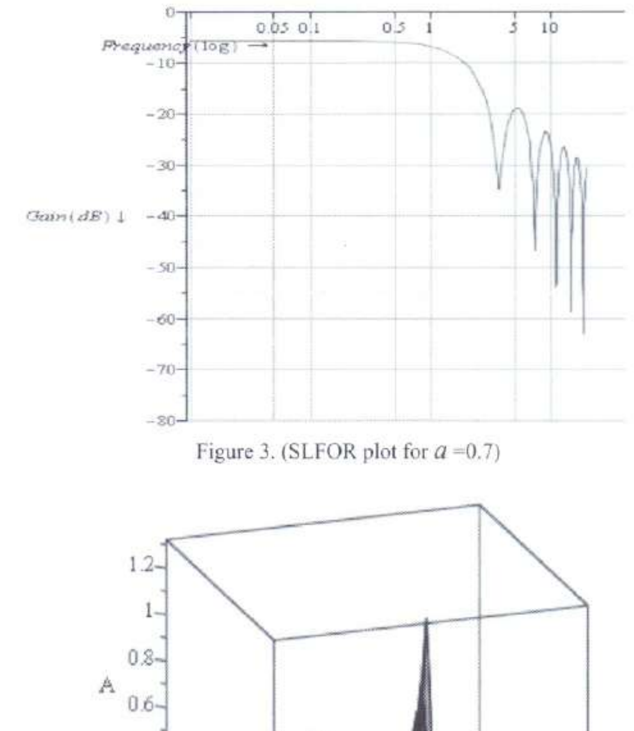

Figure 3. (SLFOR plot for a =0.7)

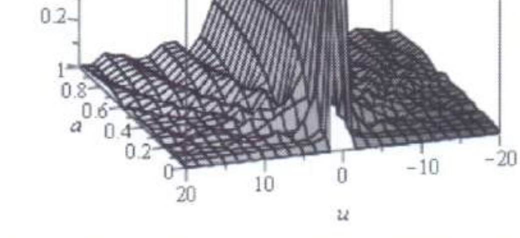

Figure 4. (3- dim. Continuous values of FRFT Dirichlet window)

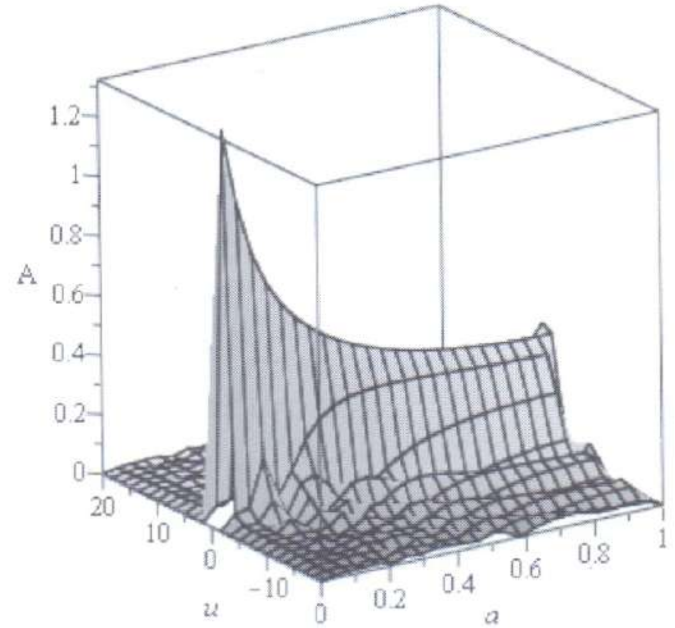

Figure 5. (3- dim. Continuous values of FRFT Dirichlet window)

TABLE I
Parameters of Dirichlet window functions

S.No.	a	Peak of main-lobe	MSLL (dB)	HMLW (normalized)	SLFOR (dB/octave)	-3dB BW (normalized)	-6dB BW (normalized)
1	0.99	-8.178	-13.13	.25295	-16.89	.13825	.18825
2	0.9	-8.102	-13.08	0.25	-13.92	.13825	.18825
3	0.8	-7.923	-13.16	.2412	-14.46	.13235	.1794
4	0.7	-7.63	-13.24	0.22645	-19.97	.12645	.16765
5	0.6	-7.205	-13.21	.20295	-14.78	.1147	.15295
6	0.5	-6.628	-13.05	.1794	-15.05	.09705	.13235
7	0.4	-5.997	-13.01	.14705	-18.49	.08235	.11175
8	0.3	-4.731	-13.06	.11175	-14.17	.06175	.0853
9	0.2	-3.1	-12.9	.07645	-12.97	.0412	.0588
10	0.1	-0.342	-11.89	0.03825	-14.67	.0206	.0294

V. CONCLUSION

We have presented a closed form solution of Dirichlet window function in fractional domain. This will help the signal processing community for spectrum shaping and real-time processing of non-stationary signals. We can see from the graphs that as the value of the tunable parameter is increased from 0.1 to 0.99, side lobe levels are reducing from -11.89 dB to -13.13 dB down from the main peak which has the effect of broadening of the pulse as also is evident from the -3dB and -6dB bandwidth values. The frequency is normalized by the factor 20 so as to calculate the parameters feasibly.

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