Application of Laplace Decomposition Algorithm to Solve the System of Homogeneous Linear Partial Differential Equations

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Abstract- **In this paper, Laplace decomposition algorithm (LDA) is introduced for the approximate solution of the system of homogeneous linear partial differential equations. The technique is described and illustrated with some numerical applications. The results assert that this scheme is rapidly convergent and quite accurate by which it approximates the solution using only few terms of its iteration scheme.**

Keywords- **Laplace decomposition method, System of homogeneous linear partial differential equations, Laplace transform, Inverse Laplace transform.**

I. INTRODUCTION

System of homogeneous linear partial differential equations has attracted much attention in a variety of applied science because of their wide applicability. These systems were formally derived to describe wave propagation to model the shallow water waves[1-3] and to examine some chemical reaction –diffusion model of Brusselator[4].

In this work, we used Laplace decomposition algorithm(LDA) because this scheme provides the solution in a rapidly convergent series with components that are elegantly computed. The Laplace decomposition algorithm was first proposed by Khuri[5-6] which is further used by Yusufoglu[7] to solve Duffing equation and Elgazery[8] for Falkner-Skan equation. The modification of Laplace decomposition method introduced by Hussain and Khan[9].Khan and Gondal[12] applied Laplace decomposition method for a new analytical solution of foam drainage equation.

Khan and Hussain[13] applied Laplace decomposition method on semi-infinite domain. The restriction and improvements of Laplace decomposition method was given by Khan and Gondal[14]. Zafar et.al. used Laplace decomposition method to solve Burger's equaton[11]. It is worth mentioning that the proposed method is an elegant combination of Laplace transform and decomposition algorithm. The advantage of this proposed method is its capability of combining two powerful methods for obtaining exact solution. The aim of this work is to establish exact solutions or approximate solutions of high degree of accuracy for the system of homogeneous linear partial differential equations.

II. LAPLACE DECOMPOSITION ALGORITHM

In this section, we present Laplace decomposition algorithm(LDA)for solving the system of homogeneous linear partial differential equations written in an operator form

$$D_t u + K_1(u,v) = 0, \qquad (1)$$
$$D_t v + K_2(u,v) = 0$$

with initial conditions

$$u(x,0) = f(x),$$
$$v(x,0) = g(x), \qquad (2)$$

where D_t is considered a first order partial differential operator, K_1 and K_2 are linear operators.

The method consists of first applying the Laplace transform to both sides of equations in system (1), we have

$$L\{D_t u\} + L\{K_1(u,v)\} = 0$$
$$L\{D_t v\} + L\{K_2(u,v)\} = 0 \qquad (3)$$

Using the differentiation property of Laplace transform and initial conditions (2) in (3), we have

$$L\{u\} = f(x)/s - 1/s\, L\{K_1(u,v)\} \qquad (4)$$
$$L\{v\} = g(x)/s - 1/s\, L\{K_2(u,v)\}$$

Operating inverse Laplace transform on both sides of (4), we have

$$u(x,t) = f(x) - L^{-1}\{1/s\, L\{K_1(u,v)\}\}$$
$$v(x,t) = g(x) - L^{-1}\{1/s\, L\{K_2(u,v)\}\} \qquad (5)$$

The Laplace decomposition algorithm(LDA) defines the solutions u(x,t) and v(x,t) by the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n \text{ and } v(x,t) = \sum_{n=0}^{\infty} v_n \qquad (6)$$

Substituting (6) in (5), we have

$$\sum_{n=0}^{\infty} u_n = f(x) - L^{-1}\{1/s\, L\{K_1(\sum_{n=0}^{\infty} u_n, \sum_{n=0}^{\infty} v_n)\}\}$$
$$\sum_{n=0}^{\infty} v_n = g(x) - L^{-1}\{1/s\, L\{K_1(\sum_{n=0}^{\infty} u_n, \sum_{n=0}^{\infty} v_n)\}\} \qquad (7)$$

In general, the recursive relation is given by:

$u_0 = f(x)$ and $v_0 = g(x)$ (8)

$u_{n+1}(x,t) = -L^{-1}\{ 1/s\, L\{ K_1(u_n,v_n)\}\}, n\geq 0$

and (9)

$v_{n+1}(x,t) = -L^{-1}\{ 1/s\, L\{ K_2(u_n,v_n)\}\}, n\geq 0$

III. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Laplace decomposition algorithm (LDA) to solve the system of homogeneous linear partial differential equations.

A. APPLICATION:1

Consider the system of homogeneous linear partial differential equations

$u_t+v_x-(u-v)=0$

$v_t - u_x + (u-v) = 0$ (10)

with initial conditions

$u(x,0)=e^x, v(x,0)=e^{-x}$ (11)

Taking the Laplace transform on both sides of (10) and using the differentiation property of Laplace transform and initial conditions (11), we have

$L\{u\} = e^x/s - 1/s\, L\{v_x\} + 1/s\, L\{u-v\}$ (12)

$L\{v\} = e^{-x}/s + 1/s\, L\{u_x\} - 1/s\, L\{u-v\}$

Operating inverse Laplace transform on both sides of (12), we have

$u(x,t) = e^x - L^{-1}\{1/s\, L\{v_x\}\} + L^{-1}\{1/s\, L\{u-v\}\}$

$v(x,t) = e^{-x} + L^{-1}\{1/s\, L\{u_x\}\} - L^{-1}\{1/s\, L\{u-v\}\}$ (13)

Then, by using the Laplace decomposition algorithm(LDA) which defines the solutions u(x ,t) and v(x ,t) by the infinite series as:

$u(x,t) = \Sigma_{n=0}^{\infty} u_n$ and $v(x,t) = \Sigma_{n=0}^{\infty} v_n$ (14)

and the terms u_x and v_x by the infinite series

$u_x(x,t) = \Sigma_{n=0}^{\infty} u_{nx}$ and $v_x(x,t) = \Sigma_{n=0}^{\infty} v_{nx}$ (15)

Substituting (14) and (15) in (13), we have

$\Sigma_{n=0}^{\infty} u_n = e^x - L^{-1}\{1/s\, L\{ \Sigma_{n=0}^{\infty} v_{nx} \}\} + L^{-1}\{1/s\, L\{\Sigma_{n=0}^{\infty} u_n - \Sigma_{n=0}^{\infty} v_n\}\}$ (16)

$\Sigma_{n=0}^{\infty} v_n = e^{-x} + L^{-1}\{1/s\, L\{\Sigma_{n=0}^{\infty} u_{nx} \}\} - L^{-1}\{1/s\, L\{\Sigma_{n=0}^{\infty} u_n - \Sigma_{n=0}^{\infty} v_n\}\}$

From (16) our required recursive relation is given by:

$u_0 = e^x$ and

$v_0 = e^{-x}$ (17)

$u_{n+1}(x,t) = - L^{-1}\{1/s\, L\{v_{nx}\}\} + L^{-1}\{1/s\, L\{u_n-v_n\}\}$, $n\geq 0$ and

$v_{n+1}(x,t) = L^{-1}\{1/s\, L\{u_{nx}\}\} - L^{-1}\{1/s\, L\{u_n-v_n\}\}$, $n\geq 0$ and (18)

The first few components of $u_n(x,t)$ and $v_n(x,t)$ by using recursive relation (18) as follows immediately

$u_1=te^x$ and $v_1= te^{-x}$ (19)

$u_2 = e^x t^2/2!$ and $v_2 = e^{-x}t^2/2!$ (20)

and so on for other components. Using (14), the series solutions are therefore given by (21)

$u(x,t) = e^x(1 + t + t^2/2! + t^3/3! + \ldots\ldots),$

$v(x,t) = e^{-x} (1 + t + t^2/2! + t3/3! \ldots\ldots)$

that converges to the exact solutions

$u(x,t) = e^{x+t}$

$v(x,t) = e^{-x+t}$ (22)

B. APPLICATION:2

Consider the system of homogeneous linear partial differential equations (23)

$u_t - v_x + (u+v) = 0$

$v_t - u_x + (u+v) = 0$

with initial conditions

$u(x,0) = \sinh x, v(x,0) = \cosh x$ (24)

Taking the Laplace transform on both sides of (23) then, by using the differentiation property of Laplace transform and initial conditions (24), we have

$L\{u\} = \sinh x/s + 1/s\, L\{v_x\} - 1/s\, L\{u+v\}$

$L\{v\} = \cosh x/s + 1/s\, L\{u_x\} - 1/s\, L\{u+v\}$ (25)

Operating inverse Laplace transform on both sides of (25), we have

$u(x,t) = \sinh x + L^{-1}\{1/s\, L\{v_x\}\} - L^{-1}\{1/s\, L\{u+v\}\}$

$v(x,t) = \cosh x + L^{-1}\{1/s\, L\{u_x\}\} - L^{-1}\{1/s\, L\{u+v\}\}$ (26)

The Laplace decomposition algorithm(LDA) defines the solutions u(x ,t) and v(x ,t) by the infinite series

$u(x,t) = \Sigma_{n=0}^{\infty} u_n$ and $v(x,t) = \Sigma_{n=0}^{\infty} v_n$ (27)

and the terms u_x and v_x by the infinite series

$u_x(x,t) = \Sigma_{n=0}^{\infty} u_{nx}$ and $v_x(x,t) = \Sigma_{n=0}^{\infty} v_{nx}$ (28)

Substituting (27) and (28) in (26), we have

$\Sigma_{n=0}^{\infty} u_n = \sinh x + L^{-1}\{1/s\, L\{ \Sigma_{n=0}^{\infty} v_{nx}\}\} - L^{-1}\{1/s\, L\{\Sigma_{n=0}^{\infty} u_n + \Sigma_{n=0}^{\infty} v_n\}\}$

$\Sigma_{n=0}^{\infty} v_n = \cosh x + L^{-1}\{1/s\, L\{\Sigma_{n=0}^{\infty} u_{nx}\}\} - L^{-1}\{1/s\, L\{\Sigma_{n=0}^{\infty} u_n + \Sigma_{n=0}^{\infty} v_n\}\}$ (29)

From (29) our required recursive relation is given by:

$u_0 = \sinh x$ and $v_0 = \cosh x$ (30)

$u_{n+1}(x,t) = L^{-1}\{1/s\, L\{v_{nx}\}\} - L^{-1}\{1/s\, L\{u_n+v_n\}\}, n\geq 0$

and

$v_{n+1}(x,t) = L^{-1}\{1/s\, L\{u_{nx}\}\} - L^{-1}\{1/s\, L\{u_n+v_n\}\}, n\geq 0$ (31)

The first few components of un(x,t) and vn(x,t) by using recursive relation (31) as follows immediately

$u_1 = -t\cosh x$ and $v_1 = -t\sinh x$ (32)

$u_2 = t^2/2!\sinh x$ and $v_2 = t^2/2!\cosh x$ (33)

$u_3 = -t^3/3!\cosh x$ and $v_3 = -t^3/3!\sinh x$ (34)

and so on for other components. Using (27), the series solutions are therefore given by

$$u(x,t) = \sinh x(1 + t^2/2! + t^4/4! + \ldots\ldots) - \cosh x(t + t^3/3! + t^5/5! + \ldots\ldots\ldots\ldots), \quad (35)$$

$$v(x,t) = \cosh x(1 + t^2/2! + t^4/4! + \ldots\ldots) - \sinh x(t + t^3/3! + t^5/5! + \ldots\ldots\ldots\ldots)$$

that converges to the exact solutions

$u(x,t) = \sinh(x\text{-}t)$ and $v(x,t) = \cosh(x\text{-}t)$ (36)

IV. CONCLUSION

In this paper, we have successfully developed the Laplace decomposition algorithm (LDA) for the solution of the system of homogeneous linear partial differential equations. The given application showed that the exact solution have been obtained even with just the two (application:1) or three (application:2) first terms of the LDA solution, which indicates that the proposed method LDA need much less computational work. The proposed scheme can be applied for the system more than two homogeneous linear partial differential equations.

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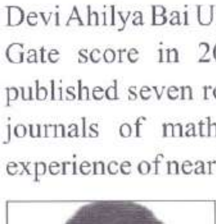

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