Application of Laplace Decomposition Algorithm (LDA) to Solve the System of Weakly Singular Volterra Integral Equations

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Abstract- **In this paper, the Laplace Decomposition Algorithm (LDA) is introduced for the exact solution of the system of weakly singular Volterra integral equations. The technique is described and illustrated with some numerical applications. The results assert that this scheme is rapidly convergent and give the exact result using only few terms of its iteration scheme.**

Keywords- **Laplace decomposition algorithm(LDA), System of weakly singular Volterra integral equations, Laplace transform, Convolution theorem, Inverse Laplace transform, Noise terms phenomenon.**

I. INTRODUCTION

System of weakly singular Volterra integral equations appear in many branches of scientific fields[1-6],such as electron emission, radar ranging, radio astronomy, atomic scattering, plasma diagnostics and optical fiber evaluation. Studies of systems of singular integral equations have attracted much concern in applied science.

The system of weakly singular Volterra integral equations of the convolution type in two unknowns [7] is given by

$$U(x) = f_1(x) + \int_0^x [k_{11}(x,t)U(t) + k_{12}(x,t)V(t)]dt,$$
$$U(x) = f_2(x) + \int_0^x [k_{21}(x,t)U(t) + k_{22}(x,t)V(t)]dt \quad (1)$$

The kernels $k_{ij}(x,t)$, $1\leq i, j\leq 2$ and the functions $f_i(x)$, i=1, 2 are given real-valued functions. The kernel k_{ij} are singular kernels given by

$$k_{ij} = 1/(x-t)^{\alpha_{ij}},\ 1 \leq 1,\ j \leq 2. \quad (2)$$

Recall that the kernel is called weakly singular as the singularity may be transformed away by a change of variable[2].

In this work, we used Laplace decomposition algorithm(LDA) because this scheme provides the solution in a rapidly convergent series with components that are elegantly computed. The Laplace decomposition algorithm(LDA) was first proposed by Khuri[8-9] which is further used by Yusufoglu[10] to solve Duffing equation and Elgazery[11] for Falkner-Skan equation.Wazwaz[12] established a necessary condition that is essentially needed to ensure the appearance of "noise terms" in the inhomogeneous equations. The necessary condition for the "noise terms" to appear in the components u0 and u1 is that the exact solution must appear as the part of u0 among other terms. The restriction and improvements of Laplace decomposition method was given by Khan and Gondal[13]. Zafar et.al. used Laplace decomposition method to solve Burger's equaton[14].

It is worth mentioning that the proposed method is an elegant combination of Laplace transform and decomposition algorithm. The advantage of this proposed method is its capability of combining two powerful methods for obtaining exact solution. The aim of this work is to establish exact solutions for the system of weakly singular Volterra integral equations without large computational work.

II. THE LAPLACE DECOMPOSITION ALGORITHM

In this section, we present Laplace decomposition algorithm(LDA) for solving the system of weakly singular Volterra integral equations given by (1) and (2). The method consists of first applying the Laplace transform to both sides of equations in system (1), we have

$$L\{U(x)\} = L\{f_1(x)\} + L\{\int_0^x [k_{11}(x,t)U(t) + k_{12}(x,t)V(t)]dt\},$$
$$L\{V(x)\} = L\{f_2(x)\} + L\{\int_0^x [k_{21}(x,t)U(t) + k_{22}(x,t)V(t)]dt\} \quad (3)$$

Using Convolution theorem of the Laplace transform, we have

$$L\{U(x)\}= L\{f_1(x)\}+ L\{k_{11}(x)\}L\{U(x)\}+ L\{k_{12}(x)\}L\{V(x)\},$$
$$L\{V(x)\}=L\{f_2(x)\}+L\{k_{21}(x)\}L\{U(x)\}+ L\{k_{22}(x)\}L\{V(x)\} \quad (4)$$

Operating inverse Laplace transform on both sides of (4), we have

$U(x) = f_1(x) + L^{-1}\{L\{k_{11}(x)\}L\{U(x)\}\} +$
$L^{-1}\{L\{k_{12}(x)\}L\{V(x)\}\},$
$V(x) = f_2(x) + L^{-1}\{L\{k_{21}(x)\}L\{U(x)\}\} +$
$L^{-1}\{L\{k_{22}(x)\}L\{V(x)\}\}$ (5)

The Laplace decomposition algorithm(LDA) defines the solutions U(x) and V(x) by the infinite series

$U(x) = \Sigma_{n=0}^{\infty} U_n$ and $V(x) = \Sigma_{n=0}^{\infty} v_n$ (6)

Substituting (6) in (5), we have (7)

$\Sigma_{n=0}^{\infty} U_n = f_1(x) + L^{-1}\{L\{k_{11}(x)\}L\{\Sigma_{n=0}^{\infty} U_n\}\}$
$+ L^{-1}\{L\{k_{12}(x)\}L\{\Sigma_{n=0}^{\infty} V_n\}\},$
$\Sigma_{n=0}^{\infty} V_n = f_2(x) + L^{-1}\{L\{k_{21}(x)\}L\{\Sigma_{n=0}^{\infty} U_n\}\}$
$+ L^{-1}\{L\{k_{22}(x)\}L\{\Sigma_{n=0}^{\infty} V_n\}\}$ (7)

In general, the recursive relations are given by:

$U_0(x) = f_1(x)$ and $V_0(x) = f_2(x)$ (8)

$U_{n+1}(x) = L^{-1}\{L\{k_{11}(x)\}L\{\Sigma_{n=0}^{\infty} U_n\}\}+$
$L^{-1}\{L\{k_{12}(x)\}L\{\Sigma_{n=0}^{\infty} V_n\}\}$, $n \geq 0$ and
$V_{n+1}(x) = L^{-1}\{L\{k_{21}(x)\}L\{\Sigma_{n=0}^{\infty} U_n\}\}+$
$L^{-1}\{L\{k_{22}(x)\}L\{\Sigma_{n=0}^{\infty} V_n\}\}$ $n \geq 0$. (9)

III. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Laplace decomposition algorithm (LDA) to solve the system of weakly singular Volterra integral equations.

A. APPLICATION: 1

Consider the system of weakly singular Volterra integral equations

$U(x) = 1 + x^2 - 5x^{4/5}/2 + \int_0^x [U(t)/(x-t)^{1/5}$
$+ V(t)/(x-t)^{1/5}]dt,$
$V(x) = 1 - x^2 - 5x^{4/5}/2 + \int_0^x [U(t)/(x-t)^{1/5} +$
$V(t)/(x-t)^{1/5}]dt$ (10)

Applying the Laplace transform to both sides of (10), we have

$L\{U(x)\} = 1/s + 2/s^3 - 2\Gamma(4/5)/s^{9/5} +$
$L\{\int_0^x [U(t)/(x-t)^{1/5} + V(t)/(x-t)^{1/5}]dt\},$
$L\{V(x)\} = 1/s - 2/s^3 - 2\Gamma(4/5)/s^{9/5} +$
$L\{\int_0^x [U(t)/(x-t)^{1/5} + V(t)/(x-t)^{1/5}]dt\}$ (11)

Using Convolution theorem of the Laplace transform, we have

$L\{U(x)\} = 1/s + 2/s^3 - 2\Gamma(4/5)/s^{9/5} +$
$\Gamma(4/5)/s^{4/5}L\{U(x)\} + \Gamma(4/5)/s^{4/5}L\{V(x)\},$
$L\{V(x)\} = 1/s - 2/s^3 - 2\Gamma(4/5)/s^{9/5} +$
$\Gamma(4/5)/s^{4/5}L\{U(x)\} + \Gamma(4/5)/s^{4/5}L\{V(x)\}$ (12)

Operating inverse Laplace transform on both sides of (12), we have

$U(x) = 1 + x^2 - 5x^{4/5}/2 + L^{-1}\{\Gamma(4/5)/s^{4/5}L\{U(x)\}\} +$
$L^{-1}\{\Gamma(4/5)/s^{4/5}L\{V(x)\}\},$
$V(x) = 1 - x^2 - 5x^{4/5}/2 + L^{-1}\{\Gamma(4/5)/s^{4/5}L\{U(x)\}\} +$
$L^{-1}\{\Gamma(4/5)/s^{4/5}L\{V(x)\}\}$ (13)

The Laplace decomposition algorithm (LDA) defines the solutions U(x) and V(x) by the infinite series

$U(x) = \Sigma_{n=0}^{\infty} U_n$ and $V(x) = \Sigma_{n=0}^{\infty} v_n$ (14)

Substituting (14) in (13), we have

$\Sigma_{n=0}^{\infty} U_n = 1 + x^2 - 5x^{4/5}/2 +$
$L^{-1}\{\Gamma(4/5)/s^{4/5}L\{\Sigma_{n=0}^{\infty} U_n\}\}+ L^{-1}$
$\{\Gamma(4/5)/s^{4/5}L\{\Sigma_{n=0}^{\infty} V_n\}\},$ (15)
$\Sigma_{n=0}^{\infty} V_n = 1 - x^2 - 5x^{4/5}/2 + L^{-1}$
$\{\Gamma(4/5)/s^{4/5}L\{\Sigma_{n=0}^{\infty} U_n\}\}+ L^{-1}$
$\{\Gamma(4/5)/s^{4/5}L\{\Sigma_{n=0}^{\infty} V_n\}\}$

From (15) our required recursive relations are given by:

$U_0(x)=1+x^2-5x^{4/5}/2$ and $V_0(x)=1-x^2-5x^{4/5}/2$ (16)

$U_{n+1}(x) = L^{-1}\{\Gamma(4/5)/s^{4/5}L\{U_n\}\} + L^{-1}$
$\{\Gamma(4/5)/s^{4/5}L\{V_n\}\}$, $n \geq 0$ and (17)
$V_{n+1}(x) = L^{-1}\{\Gamma(4/5)/s^{4/5}L\{U_n\}\} + L^{-1}$
$\{\Gamma(4/5)/s^{4/5}L\{V_n\}\}$, $n \geq 0$.

The first few components of $U_n(x)$ and $V_n(x)$ by using recursive relation (17) as follows immediately.

$U_1(x) = 5x^{4/5}/2 - 25[\Gamma(4/5)]^2x^{8/5}/6\Gamma(3/5)$ and
$V_1(x) = 5x^{4/5}/2 - 25[\Gamma(4/5)]^2x^{8/5}/6\Gamma(3/5)$ (18)

Now, by noise terms phenomena, we have

$U(x) = 1 + x^2$ and $V(x) = 1 - x^2$ (19)

Which are the exact solution and satisfy the (10).

B. APPLICATION: 2

Consider the system of weakly singular Volterra integral equations

$U(x) = x + x^2 - 25x^{7/5}/7 + \int_0^x [U(t)/(x-t)^{3/5} +$
$V(t)/(x-t)^{3/5}]dt,$
$V(x) = x - x^2 - 25x^{8/5}/12 + \int_0^x [U(t)/(x-t)^{2/5} +$
$V(t)/(x-t)^{2/5}]dt$ (20)

Applying the Laplace transform to both sides of (20), we have

$L\{U(x)\} = 1/s^2 + 2/s^3 - 2\Gamma(2/5)/s^{12/5} +$
$L\{\int_0^x [U(t)/(x-t)^{3/5} + V(t)/(x-t)^{3/5}]dt\},$
$L\{V(x)\} = 1/s^2 - 2/s^3 - 2\Gamma(3/5)/s^{13/5} +$
$L\{\int_0^x [U(t)/(x-t)^{2/5} + V(t)/(x-t)^{2/5}]dt\}$ (21)

Using Convolution theorem of the Laplace transform, we have

$L\{U(x)\} = 1/s^2 + 2/s^3 - 2\Gamma(2/5)/s^{12/5} +$
$\Gamma(2/5)/s^{2/5}L\{U(x)\} + \Gamma(2/5)/s^{2/5}L\{V(x)\},$ (22)
$L\{V(x)\} = 1/s^2 - 2/s^3 - 2\Gamma(3/5)/s^{13/5} +$
$\Gamma(3/5)/s^{3/5}L\{U(x)\} + \Gamma(3/5)/s^{3/5}L\{V(x)\}$

Operating inverse Laplace transform on both sides of (22), we have

$U(x) = x + x^2 - 25x^{7/5}/7 + L^{-1}\{\Gamma(2/5)/s^{2/5}$
$L\{U(x)\}\} + L^{-1}\{\Gamma(2/5)/s^{2/5} L\{V(x)\}\},$
$V(x) = x - x^2 - 25x^{8/5}/12 + L^{-1}\{\Gamma(3/5)/s^{3/5}$
$L\{U(x)\}\} + L^{-1}\{\Gamma(3/5)/s^{3/5}L\{V(x)\}\}$ (23)

The Laplace decomposition algorithm(LDA) defines the solutions $U(x)$ and $V(x)$ by the infinite series

$$U(x) = \Sigma_{n=0}^{\infty} U_n \text{ and } V(x) = \Sigma_{n=0}^{\infty} v_n \quad (24)$$

Substituting (24) in (23), we have

$$\Sigma_{n=0}^{\infty} U_n = x + x^2 - 25x^{7/5}/7 + L^{-1}\{\Gamma(2/5)/s^{2/5}$$

$$L\{\Sigma_{n=0}^{\infty} U_n\}\} + L^{-1}\{\Gamma(2/5)/s^{2/5} L\{\Sigma_{n=0}^{\infty} V_n\}\}, \quad (25)$$

$$\Sigma_{n=0}^{\infty} V_n = x - x^2 - 25x^{8/5}/12 + L^{-1}\{\Gamma(3/5)/s^{3/5} L$$

$$\{\Sigma_{n=0}^{\infty} U_n\}\} + L^{-1}\{\Gamma(3/5)/s^{3/5} L\{\Sigma_{n=0}^{\infty} V_n\}\}$$

From (25) our required recursive relations are given by:

$$U_0(x) = x + x^2 - 25x^{7/5}/7 \quad \text{and}$$

$$V_0(x) = x - x^2 - 25x^{8/5}/12 \quad (26)$$

$$U_{n+1}(x) = L^{-1}\{\Gamma(2/5)/s^{2/5} L\{U_n\}\} +$$

$$L^{-1}\{\Gamma(4/5)/s^{4/5} L\{V_n\}\}, n \geq 0 \text{ and}$$

$$V_{n+1}(x) = L^{-1}\{\Gamma(4/5)/s^{4/5} L\{U_n\}\} +$$

$$L^{-1}\{\Gamma(4/5)/s^{4/5} L\{V_n\}\}, n \geq 0. \quad (27)$$

The first few components of $U_n(x)$ and $V_n(x)$ by using recursive relation (27) as follows immediately

$$U_1(x) = 25x^{7/5}/7 - 25[\Gamma(2/5)]^2 x^{9/5}/18\Gamma(4/5)$$

$$- \Gamma(2/5)\Gamma(3/5)x^2 \text{ and}$$

$$V_1(x) = 25x^{8/5}/12 - 125[\Gamma(3/5)]^2 x^{11/5}/33\Gamma$$

$$(1/5) - \Gamma(2/5)\Gamma(3/5)x^2 \quad (28)$$

Now, by noise terms phenomena, we have

$$U(x) = x + x^2 \quad \text{and} \quad V(x) = x - x^2 \quad (29)$$

which are the exact solution and satisfy the (20).

IV. CONCLUSION

In this paper, we have successfully developed the Laplace decomposition algorithm (LDA) for the solution of the system of weakly singular Volterra integral equations. The given applications showed that the exact solution have been obtained even with just the two first terms of the LDA solution, which indicates that the proposed method LDA need much less computational work. The proposed scheme can be applied for the system more than two weakly singular Volterra integral equations.

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